Lecture 4: let Y be a scheme/stack; its algebraic K-theory is the abelian group.

$$K(Y) = \mathbb{Z}\text{-span}\left\{\text{coherent sheaves on } Y\right\} \Big/ \begin{array}{l}[A]+[C]=[B] \text{ if } \exists \text{ s.e.s.} \\ 0\to A\to B\to C\to 0.\end{array}$$

- for any morphism $\pi : Y \to Z$, we have
  - (derived) push-forward $\quad \pi_* : K(Y)_{\substack{\text{proper}\\ \text{support}/Z}} \longrightarrow K(Z)$
  - (derived) pull-back $\quad \pi^* : K(Z)_{\substack{\text{finite flat ampl.} \\ \cancel{\text{for independent}} \\ \text{w.r.t. } \pi}} \longrightarrow K(Y)$

$\Big\}$ the point is $\exists$ technicalities to check

- Moreover, derived tensor product gives us an operation

$$K(Y) \otimes K(Y)_{\text{perfect}} \longrightarrow K(Y)$$

which makes $K(Y)$ into a ring if Y is nice enough (e.g. smooth)

Assume now that we have an algebraic group action $G \curvearrowright Y$

$$K_G(Y) = \mathbb{Z}\text{-span}\left\{G\text{-equivariant coherent sheaves}\right\} \Big/ [A]+[C]=[B]$$

equivariant algebraic K-theory

→ We will consider linear reductive algebraic groups G (i.e. subvarieties of GL_N)

for such G, we have $K_G(Y) \hookrightarrow K_T(Y)^W$

Weyl invariants

maximal torus

and the inclusion is actually an isomorphism for many Y's (e.g. this holds if the so-called Weyl denominator is not a zero-divisor in $K_T(Y)$)

Example: $K_G(\bullet) = \left\{\text{representations of } G\right\} \Big/ [A]+[C]=[B] = \left\{\text{characters of } G\right\}$

and the fact that $\left\{\text{characters of } G\right\} = \left\{\text{characters of } T\right\}^W$ is well-known

Example: $K_{GL_N}(\cdot)$ is the polynomial ring generated by $e_i = \mathrm{Tr}(\Lambda^i \mathbb{C}^N)$ and $(e_N)^{-1}$

$K_{T_N}(\cdot) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$ with x_i being the i-th elementary character of $T \cong (\mathbb{C}^*)^N$

The identification $K_{GL_N}(\cdot) = K_{T_N}(\cdot)^{S_N}$ is given by $e_i = \sum_{1 \le k_1 < \dots < k_i \le N} x_{k_1} \cdots x_{k_i}$,

In general, the pull-back map (w.r.t. $\pi : Y \to \cdot$) gives $\pi^* : K_G(\cdot) \to K_G(Y)$, which makes every $K_G(Y)$ into a module over $K_G(\cdot) = \underbrace{\mathbb{Z}[x_1^{\pm 1}, \dots, x_N^{\pm 1}]^{S_N}}_{\text{if } G = GL_N}$

Finally, we have $K(\underbrace{Y/G}_{\text{quotient stack}}) = K_G(Y)$

Suppose we have an algebraic group action $G \curvearrowright S$ a scheme

$$T^*(S/G) \cong \mu^{-1}(0)/G \text{, where } \mu: T^*S \longrightarrow (\mathrm{Lie}\, G)^* \text{ is dual to}$$

induced action

$$\mathrm{Lie}\, G \longrightarrow \mathrm{Vect}(S) \text{ induced by action}$$

Prop: when $S = \prod_{e = \overrightarrow{ij}} \mathrm{Hom}(\mathbb{A}^{n_i}, \mathbb{A}^{n_j})$ with the usual action of $G = \prod_{i \in I} GL_{n_i}$, we have

$$\prod_{e=\overrightarrow{ij}} \mathrm{Hom}(\mathbb{A}^{n_i}, \mathbb{A}^{n_j}) \times \mathrm{Hom}(\mathbb{A}^{n_j}, \mathbb{A}^{n_i}) \xrightarrow{\mu} \prod_{i \in I} \mathrm{Hom}(\mathbb{A}^{n_i}, \mathbb{A}^{n_i}) = (\mathrm{Lie}\, G)^*$$

$$(\phi_e \quad , \quad \phi_e^*) \rightsquigarrow \sum_e (\phi_e \phi_e^* - \phi_e^* \phi_e)$$

$$\Rightarrow T^* \mathrm{Rep}_n Q = \left\{ (\phi_e : \mathbb{A}^{n_i} \rightleftarrows \mathbb{A}^{n_j} : \phi_e^*)_{e = \overrightarrow{ij}} \text{ s.t. } \sum_e (\phi_e \phi_e^* - \phi_e^* \phi_e) = 0 \right\} / G$$

$\mathbb{C}^*$ acts as follows: if there are d arrows between vertices i and j, then the ϕ_e and ϕ_e^* provide d maps from $\mathbb{A}^{n_i}$ to $\mathbb{A}^{n_j}$ and d maps from $\mathbb{A}^{n_j}$ to $\mathbb{A}^{n_i}$; we require $t \in \mathbb{C}^*$ to scale the former maps by $(t^d, t^{d-2}, \dots, t^{2-d})$ and the latter maps by $(t^{2-d}, \dots, t^{d-2}, t^d)$; this way, any composition $\phi_e \phi_e^*$ and $\phi_e^* \phi_e$ will be scaled by t^2

Define
$$\widetilde{\mathcal{E}xt}_{n,n'} Q = \left\{ (\phi_e : \mathbb{A}^{n_i + n_i'} \rightleftarrows \mathbb{A}^{n_j + n_j'} : \phi_e^*)_{e = \overrightarrow{ij}} \text{ s.t. } \begin{array}{l} \sum_e (\phi_e \phi_e^* - \phi_e^* \phi_e) = 0 \\ \phi_e(\mathbb{A}^{n_i}) \subset \mathbb{A}^{n_j}, \ \phi_e^*(\mathbb{A}^{n_j}) \subset \mathbb{A}^{n_i} \end{array} \right\} \Big/ \begin{array}{l} (g_i \in GL_{n_i + n_i'}) \\ \text{s.t. } g_i(\mathbb{A}^{n_i}) \subset \mathbb{A}^{n_i} \end{array}$$

$$\begin{tikzcd} & \widetilde{\mathcal{E}xt}_{n,n'} Q \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ T^*\mathrm{Rep}_{n'} Q \times T^* \mathrm{Rep}_n Q & & T^* \mathrm{Rep}_{n+n'} Q \end{tikzcd}$$

Def (K-theoretic Hall algebra): $\boxed{K\text{-}HA_Q = \bigoplus_{n \in \mathbb{N}^I} K_{\mathbb{C}^*}(T^* \mathrm{Rep}_n Q)}$ with multiplication

$p_{2*}(\mathcal{L}_{n,n'} \otimes p_1^*)$, where $\mathcal{L}_{n,n'}$ are line bundles on $\widetilde{\mathcal{E}xt}_{n,n'} Q$ for associativity

quite involved thing to prove, especially since the pull-back p_1^* needs to be "refined" appropriately

explicitly, $K_{\mathbb{C}^*}(T^* \mathrm{Rep}_n Q) = K_{\mathbb{C}^* \times G}(\mu^{-1}(0))$

$$\downarrow i_*$$

$$K_{\mathbb{C}^* \times G}(\text{affine space}) = K_{\mathbb{C}^* \times G}(\bullet) = \mathbb{Z}[q^{\pm 1}][z_{i1}^{\pm 1}, \dots, z_{in_i}^{\pm 1}]^{\text{symmetric}}_{i \in I}$$

and so we obtain an abelian group homomorphism

$$K\text{-}HA_Q \xrightarrow{i_*} \mathcal{V}_{\text{int}} = \bigoplus_{n \in \mathbb{N}^I} \mathbb{Z}[q^{\pm 1}][z_{i1}^{\pm 1}, \dots, z_{in_i}^{\pm 1}]^{\text{symmetric}}_{i \in I}$$

Prop: i_* intertwines the multiplication on K-HA$_Q$ with the following shuffle product on $\mathcal{V}_{\text{int}}$

$$R(z_{i1}, \dots, z_{in_i})_{i \in I} * R'(z_{i1}, \dots, z_{in_i'}) = \mathrm{Sym}\left[\frac{R(z_{i1}, \dots, z_{in_i}) R'(z_{i,n_i+1}, \dots, z_{i,n_i+n_i'})}{\prod_{i \in I} n_i! \ \prod_{i \in I} n_i'!} \prod_{i,j \in I} \prod_{\substack{a \le n_i \\ b > n_j}} \zeta_{ij}\left(\frac{z_{ia}}{z_{jb}}\right) \right]$$

where $\zeta_{ii}(x) = \frac{1 - xq^2}{1 - x}$, $\zeta_{ij}(x) = (1 - xq^{-d})(1 - xq^{-d+2}) \dots (1 - xq^{d-2})$ · some monomial in x, q that depends on $L_{n,n'}$

total edges between i and j

Let us perform a sanity check on the ring homomorphism i_*; since

$$\mathcal{V} : U_q^+(\mathfrak{g}) \xrightarrow{\text{expected}} K\text{-}HA_Q \xrightarrow{i_*} \mathcal{V}_{int}$$

should be algebra homomorphisms, let us check that the relations between the generators $e_i \in U_q^+(\mathfrak{g})$ are respected in $\mathcal{V}_{int}$

$\hookrightarrow$ since $K_{\mathbb{C}^*}(T^*Rep_{(0,0,\dots,0,1,0,\dots,0)}) = K_{\mathbb{C}^* \times GL_1}(\text{affine space}) = \mathbb{Z}[q^{\pm 1}][z_{i1}^{\pm 1}]$ (i-th spot)

it is natural to postulate $\boxed{\mathcal{V}(e_i) = z_{i1}^0}$

$\hookrightarrow$ lying in direct sumand $n = (0,\dots,0,1,0,\dots,0)$ of $\mathcal{V}_{int}$

Note that $\forall i \neq j \in I$ with d edges between them, we have

$$\zeta_{ij}(x) = (1 - xq^{-d}) \cdot \rho_{ij}(x) \cdot \text{(some monomial in } x)$$

where $\rho_{ij}(x) = \rho_{ji}\left(\frac{1}{x}\right)$; postulate: fix a total order on I and set

$$\zeta_{ij}(x) = (1 - xq^{-d})\, \rho_{ij}(x) \cdot \begin{cases} 1 & \text{if } i < j \\ (-x)^{-1} & \text{if } i > j \end{cases}$$

more generally this could be $\gamma_{ij} x^{a_{ij}}$ and this would have to be $\gamma_{ij} (-x)^{-1-a_{ij}}$

Prop: $\forall i \neq j \in I$, we have $\mathcal{V}\left(\sum_{k=0}^{d+1} (-1)^k \binom{d+1}{k}_q e_i^k e_j e_i^{d+1-k}\right) = 0$, i.e.

$$\mathrm{Sym}\left[\prod_{1 \le a < b \le d+1} \frac{z_a q^2 - z_b}{z_a - z_b} \cdot \sum_{k=0}^{d+1} (-1)^k \binom{d+1}{k}_q \cdot \prod_{a=1}^{k} \left(1 - \frac{z_a}{wq^d}\right) \prod_{b=k+1}^{d+1} \left(\frac{1}{q^d} - \frac{z_b}{w}\right)\right] = 0$$

in the z variables — we denoted $z_a = z_{ia}$ and $w = z_{j1}$ — assume WLOG $i < j$

The above is a combinatorial identity which you can brute force, but there is a more clever way to do it: because of the symmetrization, the poles at $z_a - z_b$ get canceled, so the LHS is a Laurent polynomial in $z_1, \dots, z_{d+1}, w$ moreover, it has total homogeneous degree 0, so we have

$$LHS = \frac{P(z_1, \dots, z_{d+1}, w)}{w^{d+1}}$$

for some polynomial P (symmetric in $z_1, \dots, z_{d+1}$); moreover, P does not have any monomials which involve some $z_a^{\ge 2}$ because the order of growth of LHS as $z_a \to \infty$ is at most linear $\Rightarrow LHS = \sum_{k=0}^{d+1} c_k \cdot \frac{\text{elem symm}_k(z_1, \dots, z_{d+1})}{w^k}$

for some constants c_k; to show that $c_k = 0$ by reverse induction on k send $\lim_{x \to \infty} LHS(xz_1, \dots, xz_k, z_{k+1}, \dots, z_{d+1})$ and use q-binomial theorem.